International Journal of Theoretical Physics, Vol. 44, No. 7, July 2005 (© 2005)
DOI: 10.1007/s10773-005-7078-3

On the (Non)Existence of States on Orthogonally Closed Subspaces in an Inner Product Space

Karl Svozil,[1,4] E. Chetcuti,[2] and P. Pták[3]

Received August 14, 2004; accepted September 2, 2004

Suppose that S is an *incomplete* inner product space. In (Dvurečenskij, 1992, Gleason's Theorem and Its Applications, Ister Science Press, Bratislava, Kluwer Academic Publishers, Dordrecht), A. Dvurečenskij shows that there are no finitely additive states on orthogonally closed subspaces, $F(S)$, of S that are regular with respect to finitely dimensional spaces. In this note we show that the most important special case of the former result—the case of the evaluations given by vectors in the "Gleason manner"—allows for a relatively simple proof. This result further reinforces the conjecture that there are no finitely additive states on $F(S)$ at all.

KEY WORDS: Hilbert space; inner product space; orthogonally closed subspace; finitely additive state.

1. INTRODUCTION

Let S be a real or complex separable inner product space and let $\langle\cdot,\cdot\rangle$ denote the inner product of S. Let us denote by $F(S)$ the set of all orthogonally closed subspaces of S. A subspace M of S is in $F(S)$ if $M = M^{\perp\perp}$, where $M^{\perp} = \{x \in S : \langle x, y\rangle = 0 \text{forall} y \in M\}$. It turns out that if we understand $F(S)$ with the ordering given by the inclusion relation and with orthocomplementation relation $M \longrightarrow M^{\perp}$ as defined above, then $F(S)$ becomes a complete orthocomplemented lattice. However, $F(S)$ does not have to be orthomodular. In fact, Amemiya and Araki (1966) proved the following algebraic criterion for the (topological) completeness of an inner product space S: an inner product space S is complete if and only if $F(S)$ is orthomodular.

[1] University of Technology, Vienna, Austria.
[2] Department of Mathematics, University of Malta, Msida MSD 06, Malta.
[3] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, 166 27 Prague 6, Czech Republic.
[4] To whom correspondence should be addressed at the University of Technology, Vienna, Austria; e-mail: svozil@tuwien.ac.at

Let us now turn to measure-theoretic criteria for the completeness of S. The following result by Hamhalter and Pták (1987) initiated a series of interesting measure theoretic characterizations for the completeness of an inner product space S (Dvurečenskij, 1992).

Theorem 1.1. *An inner product space S is complete if and only if* $F(S)$ *possesses a* σ*-additive state.*

In Pták (1988), Pták asked whether S has to be complete if $F(S)$ possesses a finitely additive state. Recently, Dvurečenskij and Pták (submitted) proved that if S is an incomplete inner product space, then the assumption that there is a finitely additive state on $F(S)$ implies that the range of this state has to be the entire interval [0, 1]. In this note we show that an inner product space S is complete if, and only if, there exists $u \in \bar{S}$ such that s_u defines a state on $F(S)$, where by $\bar{S}$ is denoted the completion of S. Here, for any vector $u \in S$ with $\|u\| = 1$, by s_u is meant the "Gleason" assignment defined by

$$\begin{aligned} s_u : F(S) &\to [0, 1] \\ M &\mapsto \langle P_{\bar{M}} u, u \rangle. \end{aligned}$$

Before we launch on the proof proper, let us summarize the "state of the art" of the state problem for $F(S)$. If there are states on $F(S)$ then there are pure states on $F(S)$ (Krein–Milman). But in view of the previous two facts these pure states must be rather bizarre. Thus, a conjecture remains that for an incomplete space S the lattice $F(S)$ is stateless.

2. RESULTS

Let S be a separable inner product space and let $\bar{S}$ be its completion. In this section we mainly prove the result formulated in the introduction.

Theorem 2.1. *A separable inner product space S is complete if, and only if, there exists* $u \in \bar{S}$ *such that*

$$s_u : M \mapsto \langle P_{\bar{M}} u, u \rangle$$

defines a state on $F(S)$.

Proof: If S is complete then, obviously, for every $u \in S = \bar{S}$, s_u is a (σ-additive) state on $F(S)$ ($F(S) = L(S)$ and this follows from Gleason's theorem).

For the second implication, suppose that there exists a vector $u \in \bar{S}$ such that s_u is a state on $F(S)$. We could be of certain importance in their own right. □

Claim 1. Suppose that there exists $u \in S$ such that

$$s_u : M \mapsto \langle P_{\bar{M}} u, u \rangle$$

defines a state on $F(s)$. Then for every unit vector $v \in S$, s_v defines a state on $F(S)$.

Proof: Let $\tilde{S}$ be a subspace of $\bar{S}$ generated by s and u. Let $v(\neq u)$ be a unit vector in S and put $P = [u] + [v]$. Then $\tilde{S} = P \oplus P^{\perp}$. Set $\tilde{w} = v - \langle v, u \rangle u$ and let $w = \frac{\tilde{w}}{\|\tilde{w}\|}$. Similarly, let $\tilde{z} = u - \langle u, v \rangle v$ and put $z = \frac{\tilde{z}}{\|\tilde{z}\|}$. Then $P = [v] \oplus [z] = [u] \oplus [w]$. Define the map

$$T : \tilde{S} \to \tilde{S}$$

$$P \oplus P^{\perp} \to \tilde{S}$$

$$p + p' = \alpha v + \beta z + p' \mapsto \alpha u + \beta w + p'.$$

T is a unitary operator on $\tilde{S}$, that is T is a bijective linear operator satisfying

$$\langle x, y \rangle = \langle Tx, Ty \rangle$$

for all $x, y \in \tilde{S}$.

By the continuity of T we can extend it over $\bar{S}$. With a harmless abuse of notation let us denote the extension again by T. We now show that if A is a subspace of S, then $\overline{TA} = T\bar{A}$. Since T is continuous it follows immediately that $T\bar{A} \subset \overline{TA}$. Let $x \in \overline{TA}$. Then $x = \lim_{i \to \infty} x_i$ where $x_i \in TA$ for all $i \in N$. Let $y_i \in A$ be such that $Ty_i = x_i$. Then we have

$$\begin{aligned} \|x_i - x_j\|^2 &= \langle Ty_i - Ty_j, Ty_i - Ty_j \rangle \\ &= \langle T(y_i - y_j), T(y_i - y_j) \rangle \\ &= \langle y_i - y_j, y_i - y_j \rangle \\ &= \|y_i - y_j\|^2. \end{aligned}$$

This implies that $\{y_i\}$ is Cauchy and therefore it converges to some $y \in \bar{A}$. That $Ty = x$ follows again by the continuity of T.

We now show that for any $A \in F(S)$, we have

$$\|P_{\bar{A}} v\|^2 = \|P_{\overline{TA}} u\|^2.$$

Let $\{a_i\} \subset A$ be an ONB of $\bar{A}$. Then $\{Ta_i\}$ is an ONB of $T\bar{A}$ $(= \overline{TA})$ in TA. We then have

$$a_i = \alpha_i v + \beta_i z + p_i'$$

and therefore

$$Ta_i = \alpha_i u + \beta_i w + p_i'.$$

This implies that

$$\begin{aligned}\|P_{\overline{A}}v\|^2 &= \sum |\langle a_i, v\rangle|^2 \\ &= \sum |\alpha_i|^2 \\ &= \sum |\langle Ta_i, u\rangle|^2 \\ &= \|P_{\overline{TA}}u\|^2.\end{aligned}$$

Thus, for any $A \in F(S)$, $s_v(A) = \|P_{\overline{A}}v\|^2 = \|P_{\overline{TA}}u\|^2 = s_u(TA)$, and therefore s_v does indeed define a state on $F(S)$. □

Claim 2. Suppose that, for each $u \in S$, s_u defines a state on $F(S)$. Then for every unit vector $v \in \bar{S}$, s_v defines a state on $F(S)$.

Proof: Let $v \in \overline{S} \setminus S$. There exists a sequence $\{v_i\} \subset S$ such that $v = \lim_{i\to\infty} v_i$. For any $A \in F(S)$,

$$\begin{aligned}P_{\bar{A}}v &= P_{\overline{A}} \lim_{i\to\infty} v_i \\ &= \lim_{i\to\infty} P_{\overline{A}}v_i\end{aligned}$$

and therefore

$$s_v(A) = \lim_{i\to\infty} s_{v_i}(A).$$

It is then not difficult to check that s_v defines a state on $F(S)$ (pointwise limits of finitely additive states are finitely additive states). □

Claim 3. Let for any $v \in \bar{S}$ s_v defines a state on $F(S)$. Let M be a closed subspace of S. Then

$$M \in F(S) \text{if, and only if, } \bar{M}^{\perp_S} = \bar{M}^{\perp_{\bar{S}}}.$$

Proof: Let $M \in F(S)$. We need to show that $\bar{M^{\perp_S}} = \bar{M}^{\perp_{\bar{S}}}$. It is sufficient to prove that $M^{\perp_S} \supset \bar{M}^{\perp_{\bar{S}}}$. Let $\{n_i : i \in I_{M'}\}$ be an orthonormal basis (ONB) in $M^{\perp_S}$ of $\bar{M^{\perp_S}}$, and let $\tilde{x} \in \overline{M}^{\perp_{\overline{S}}}$ $(\tilde{x} \neq 0)$ be arbitrary. Put $x = \frac{\tilde{x}}{\|\tilde{x}\|}$. Consider the state s_x on $F(S)$.

$$1 = s_x(S) = s_x(M \vee M^{\perp_S})$$

$$
\begin{aligned}
&= s_x(M) + s_x(M^{\perp_S}) \\
&= s_x(M^{\perp_S}) \qquad \text{since } x \perp M \\
&= \sum_{i \in I_{M'}} |\langle x, n_i \rangle|^2.
\end{aligned}
$$

This implies that for all $\tilde{x} \in \overline{M}^{\perp_{\overline{S}}}$,

$$
\|\tilde{x}\|^2 = \sum_{i \in I_{M'}} |\langle \tilde{x}, n_i \rangle|^2.
$$

Therefore it follows, by Parseval's identity, that $\{n_i : i \in I_{M'}\}$ is an ONB of $\bar{M}^{\perp_{\bar{S}}}$ and hence $\overline{M^{\perp_S}} = \overline{M}^{\perp_{\overline{S}}}$.

Now we prove the converse. Suppose that $\overline{M^{\perp_S}} = \overline{M}^{\perp_{\overline{S}}}$. To reach a contradiction, assume that $M \notin F(S)$. There exists $v \in M^{\perp_S \perp_S} \setminus M$ such that $v \perp M^{\perp_S}$ and $v \notin M$. This implies that $v \perp \overline{M^{\perp_S}}$ and hence $v \in \overline{M}^{\perp_{\bar{S}} \perp_{\overline{S}}} = \overline{M}$. But this would imply that $v \in \overline{M} \cap S = M$, since M is closed in S. This is the required contradiction. □

Claim 4. Suppose that for every $u \in S$ the mapping

$$
M \mapsto \langle P_{\overline{M}} u, u \rangle
$$

defines a state on $F(S)$. Let $M \in F(S)$ and let $\{x_i\}$ be any maximal orthonormal system (MONS) in M. Then $M = \{x_i\}^{\perp_S \perp_S}$.

Proof: Let $\{m_i\} \subset M$ be an ONB of $\overline{M}$ and $\{n_i\} \subset M^{\perp_S}$ be an ONB of $\overline{M^{\perp_S}} = \overline{M}^{\perp_{\overline{S}}}$. Then $\{x_i\} \cup \{n_i\}$ is a MONS of S. This implies that

$$
\begin{aligned}
M \vee M^{\perp_S} &= \vee[m_i] \bigvee \vee[n_i] \\
&= S \\
&= \vee[x_i] \bigvee \vee[n_i] \\
&= \{x_i\}^{\perp_S \perp_S} \vee M^{\perp_S}.
\end{aligned}
$$

Certainly, we have $\{x_i\}^{\perp_S \perp_S} \subset M$. Take any unit vector $y \in \overline{M}$ and consider the state s_y. We have

$$
\begin{aligned}
1 = s_y(S) &= s_y(\{x_i\}^{\perp_S \perp_S} \vee M^{\perp_S}) \\
&= s_y(\{x_i\}^{\perp_S \perp_S}) \\
&= \|P_{\overline{\{x_i\}^{\perp_S \perp_S}}} y\|.
\end{aligned}
$$

This implies that $y \in \overline{\{x_i\}^{\perp_s \perp_s}}$ and therefore

$$\overline{M} = \overline{\{x_i\}^{\perp_s \perp_s}}$$

which yields

$$M = \{x_i\}^{\perp_s \perp_s}.$$

□

Claim 5. $F(S)$ is orthomodular.

Proof: Let $A \subset B$ be in $F(S)$. Let $\{a_i\} \subset A$ be an ONB of $\overline{A}$. Extend $\{a_i\}$ to a MONS $\{a_i\} \cup \{b_i\}$ of B. It is not difficult to see that $\{b_i\}$ is a MONS in $A^{\perp_s} \cap B$ and that therefore

$$\begin{aligned} A \vee (A^{\perp_s} \wedge B) &= \{a_i\}^{\perp_s \perp_s} \vee \{b_i\}^{\perp_s \perp_s} \\ &= \vee[a_i] \bigvee \vee[bi] \\ &= B \end{aligned}$$

This completes the proof of Theorem 2.7. □

ACKNOWLEDGMENTS

P. Pták acknowledges the support of the Center of Machine Perception, Czech Technical University and the grant no. 201/02/1540 of the Grant Agency of the Czech Republic. The support enabled the P. Pták to work with E. Chetcuti on the problem studied in this paper.

REFERENCES

Amemiya, J. and Araki, H. (1996). A remark on Piron's paper. *Publications of the Research Institute for Mathematical Sciences, Kyoto University, Series A* **2**, 423–427.

Dvurečenskij, A. (1992). *Gleason's Theorem and Its Applications*, Ister Science Press, Bratislava, Kluwer Academic Publishers, Dordrecht.

Dvurečenskij, A. and Pták, P. *On states on orthogonally closed subspaces of an inner product space*, submitted for publication.

Hamhalter, J. and Pták, P. (1987), A completeness criterion for inner product spaces. *The Bulletin of the London Mathematical Society* **19**, 259–263.

Pták, P. (1988). *FAT–CAT (in the state space of quantum logics)*, Proceedings of "Winter School of Measure Theory," Liptovský Ján 1988, (Czechoslovakia).